Algebraic Geometry: Elliptic Curves and 2 Theorems

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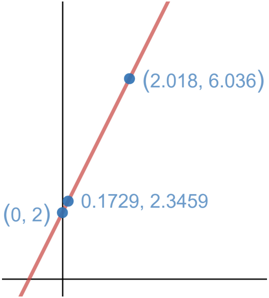

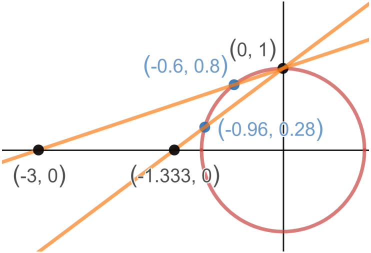

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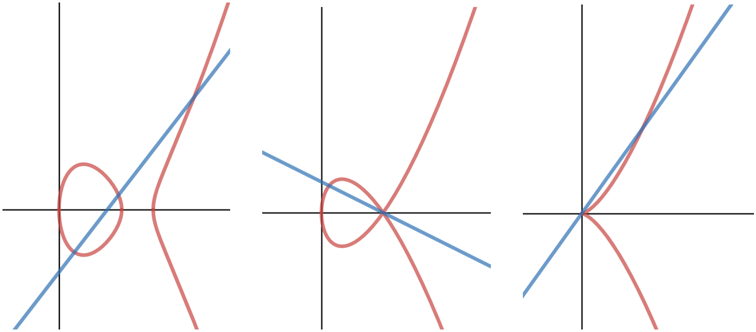
- Extending projection to degree 3:



Finding Rational Points on Elliptic Curves

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- We are very familiar with structures like $\mathbb{Z}$ which use addition
- To understand rational points on elliptic curves, can we give them similar structure?

If we can assign such a structure, finding rational points is a lot simpler:

Example.

$\mathbb{Z}$ is generated by -1 or 1; $\mathbb{Z}/7\mathbb{Z} = \{0, 1, 2, 3, 4, 5, 6\}$ is generated by anything but 0.

Instead of looking for all rational points, we can try to find a generating set.

What is a Group?

A group $(G, \circ)$ is a set G with a law of composition $(a, b) \mapsto a \circ b$ satisfying the following:

- Associativity: $(a \circ b) \circ c = a \circ (b \circ c)$
- Identity element: $\exists\, e \in G$ such that $a \circ e = e \circ a = a$
- Inverse element: for $a \in G, \exists\, a^{-1} \in G$ such that $a \circ a^{-1} = a^{-1} \circ a = e$

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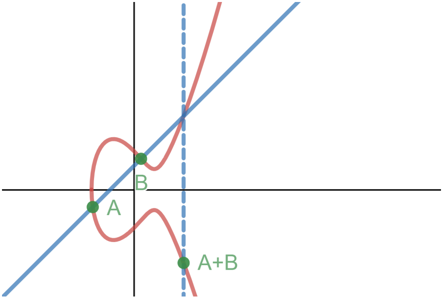
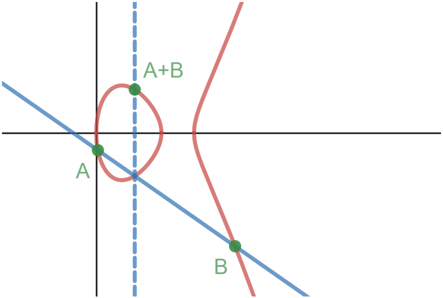
Example.

$(\mathbb{Z}, +)$ and $(\mathbb{Z}_n, +)$ are groups, as well as $(\mathrm{GL}_2(\mathbb{R}), \times)$ where

$$\mathrm{GL}_2(\mathbb{R}) = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, a, b, c, d \in \mathbb{R} \text{ and } A \text{ is invertible} \right\}.$$

If we tweak the "addition" of points mentioned before, we get a group structure for the rational points on an elliptic curve!





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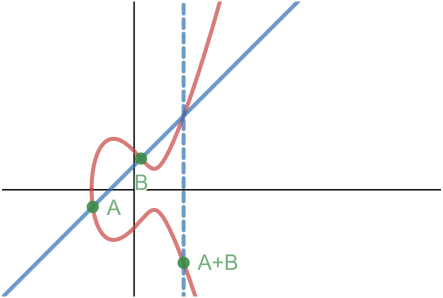

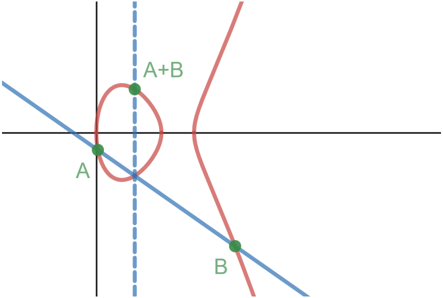

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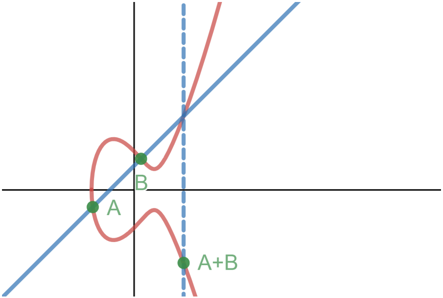

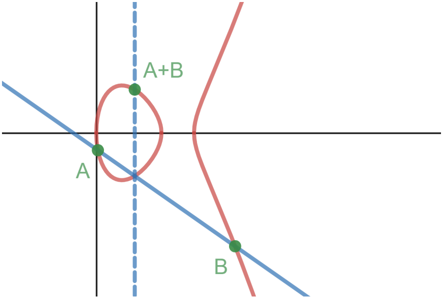

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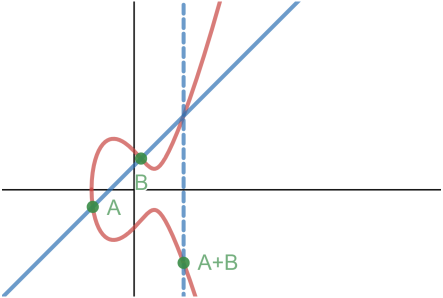

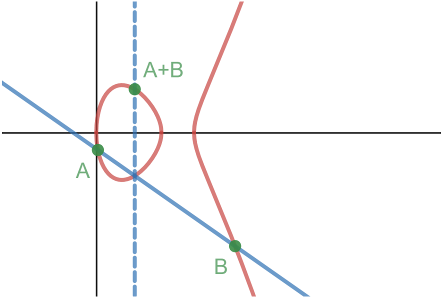

But, what about...

- Identity?
- Tangent lines?

Solution: Projective Geometry

Definition.

(Projective space) Define the equivalence relation $\sim$ *by* $(x_0, x_1, ..., x_n) \sim (y_0, y_1, ..., y_n)$ *if* $\exists\, \lambda \in k$ *such that* $y_i = \lambda x_i$*. Then, we define real projective n-space as*

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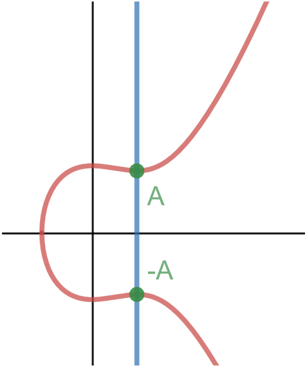
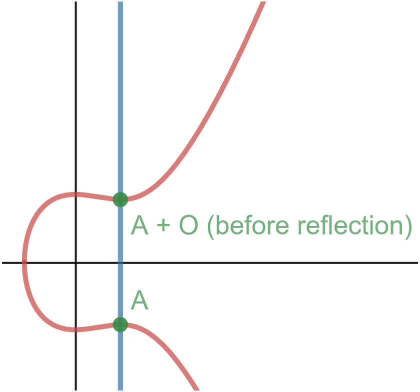
- Bézout's theorem guarantees 3 intersection points

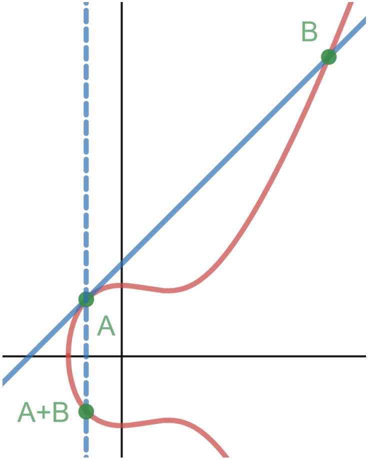
Now we can answer our questions from before about the group structure of the rational points: point at infinity on the curve, denoted $\mathcal{O}$, is the identity.





Elliptic Curve Group Structure (cont'd)

Tangent lines do have "3" intersections:



Group Properties Regarding Elliptic Curves

The group of rational points on an elliptic curve E is denoted as $E(\mathbb{Q})$.

Definition.

An element P of a group G is said to have order m if m is the minimal natural number satisfying $mP = P \circ P \circ ... \circ P$ (m times) $= e$. If no such m exists, P has infinite order.

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Definition.

The torsion subgroup of a group G is the set of all elements of G with finite order.

- Can we determine $E(\mathbb{Q})_{\text{tors}}$?

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A set $S \subset G$ for a group G is a generating set if all elements can be written as combinations of elements in S under the group operation.

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Example.

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- Can we determine the generating set for $E(\mathbb{Q})$, and is it finite or infinite?

The Nagell-Lutz Theorem

Theorem.

Let $y^2 = x^3 + ax^2 + bx + x$ be a non-singular elliptic curve with integral coefficients, and let D be the discriminant of the polynomial, $D = -4a^3c + a^2b^2 + 18abc - 4b^3 - 27c^2$. Any point (x, y) of finite order must have $x, y \in \mathbb{Z}$ and $y|D$.

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There is a stronger form of the theorem which includes $y^2|D$.

Example.

The points $\{\mathcal{O}, (1, 1), (0, 0), (1, -1)\}$ are the points of finite order on $y^2 = x^3 - x^2 + x$.

The Nagell-Lutz Theorem (cont'd)

Example.

Given a prime p, $E(\mathbb{Q})_{\text{tors}}$ for $y^2 = x^3 + px$ is always $\{\mathcal{O}, (0,0)\}$.

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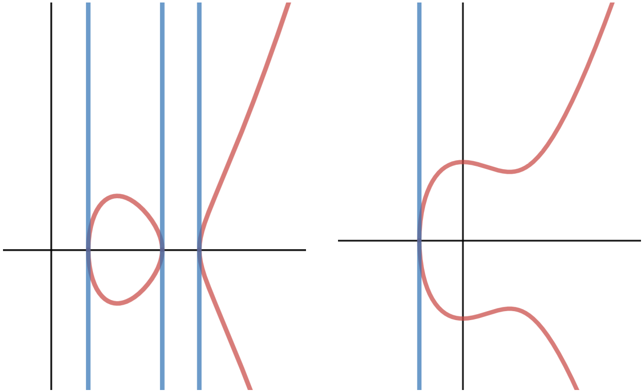

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(Mordell's Theorem) Let E be a non-singular elliptic curve with a rational point of order 2. Then $E(\mathbb{Q})$ is a finitely generated abelian group.

Any finitely generated abelian group G can be written as $\mathbb{Z}^r \oplus G_{\text{tors}}$, where r is called the rank. The rank can be computed by solving some Diophantine equations.

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Example.

Given a prime p, the rank of $y^2 = x^3 + px$ is either 0, 1, or 2.

Extras

Theorem.

(Mazur's Theorem) Let E be a non-singular cubic curve with rational coefficients, and suppose $P \in E(\mathbb{Q})$ has order m. Then either $1 \leq m \leq 10$ or $m = 12$. The only possible torsion subgroups are isomorphic to $\mathbb{Z}/N\mathbb{Z}$ for $1 \leq N \leq 10$ or $N = 12$, or $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2N\mathbb{Z}$ for $1 \leq N \leq 4$.

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Genus-degree formula: $g = \dfrac{(d-1)(d-2)}{2}$ for curves in $\mathbb{P}^2$.

Theorem.

(Falting's Theorem) A curve of genus greater than 1 has only finitely many rational points.

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